Appealing to the second part of the Fundamental Theorem, we find that

$$S'(x) = \frac{d}{dx}\int_0^x \frac{\sin t}{t}\,dt = \frac{\sin x}{x} \qquad \text{for } x > 0.$$

As anticipated, the derivative of S changes sign at integer multiples of π. Specifically, S' is positive and S increases on the intervals $(0, \pi), (2\pi, 3\pi), \ldots, (2n\pi, (2n + 1)\pi), \ldots,$ while S' is negative and S decreases on the remaining intervals. It is clear that S has local maxima at $x = \pi, 3\pi, 5\pi, \ldots,$ and it has local minima at $x = 2\pi, 4\pi, 6\pi, \ldots.$

One more observation is helpful. It can be shown that, while S oscillates for increasing x, its graph gradually flattens out and approaches a horizontal asymptote. (Finding the exact value of this horizontal asymptote is challenging; see Exercise 97.) Assembling all these observations, the graph of the sine integral function emerges (Figure 5.47b).

Related Exercises 69–72 ◂

Proof of the Fundamental Theorem Let f be continuous on $[a, b]$ and let A be the area function for f with left endpoint a. The first step is to prove that $A'(x) = f(x)$, which is Part 1 of the Fundamental Theorem. The proof of Part 2 then follows.

Step 1. We use the definition of the derivative,

$$A'(x) = \lim_{h\to 0}\frac{A(x+h) - A(x)}{h}.$$

First assume that $h > 0$. Using Figure 5.48 and Property 5 of Table 5.3, we have

$$A(x+h) - A(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt = \int_x^{x+h} f(t)\,dt.$$

That is, $A(x + h) - A(x)$ is the net area of the region bounded by the curve on the interval $[x, x + h]$.

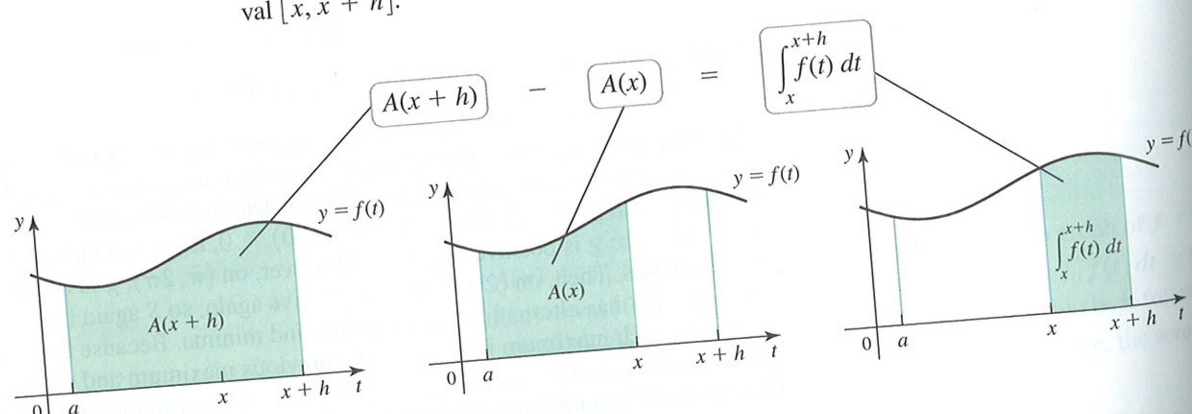


FIGURE 5.48

➤ The quantities m and M exist for any $h > 0$; however, they also depend on h. Figure 5.48 illustrates the case $0 \le m \le M$. The argument that follows holds for the general case.

Let m and M be the minimum and maximum values of f on $[x, x + h]$, respectively, which exist by the continuity of f. In the case that $0 \le m \le M$ (Figure 5.49), $A(x + h) - A(x)$ is greater than or equal to the area of a rectangle with height m and width h and it is less than or equal to the area of a rectangle with height M and width h; that is,

$$mh \le A(x+h) - A(x) \le Mh.$$

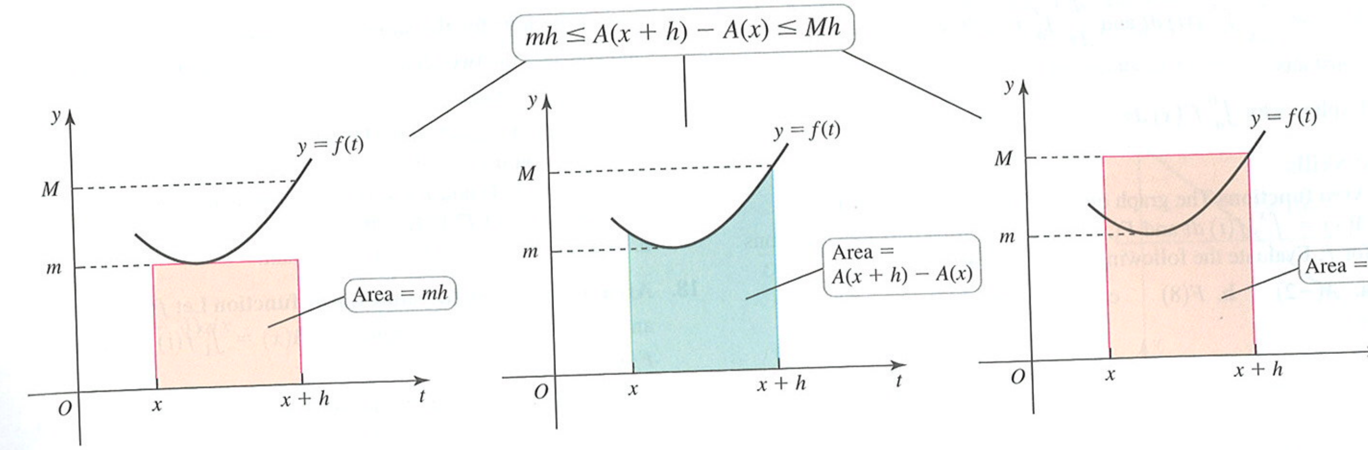


FIGURE 5.49

Dividing these inequalities by h, we have

$$m \le \frac{A(x+h) - A(x)}{h} \le M.$$

The case $h < 0$ is handled similarly and leads to the same conclusion.

We now take the limit as $h \to 0$ across these inequalities. As $h \to 0$, m and M squeeze together toward the value of $f(x)$, because f is continuous at x. At the same time, as $h \to 0$, the quotient that is sandwiched between m and M approaches $A'(x)$:

$$\underbrace{\lim_{h\to 0} m}_{f(x)} = \underbrace{\lim_{h\to 0}\frac{A(x+h) - A(x)}{h}}_{A'(x)} = \underbrace{\lim_{h\to 0} M}_{f(x)}$$

By the Squeeze Theorem (Theorem 2.5), we conclude that $A'(x) = f(x)$.

➤ Once again we use an important fact: Two antiderivatives of the same function differ by a constant.

Step 2. Having established that the area function A is an antiderivative of f, we know that $F(x) = A(x) + C$, where F is any antiderivative of f and C is a constant. Noting that $A(a) = 0$, it follows that

$$F(b) - F(a) = (A(b) + C) - (A(a) + C) = A(b).$$

Writing $A(b)$ in terms of a definite integral, we have

$$A(b) = \int_a^b f(x)\,dx = F(b) - F(a),$$

which is part 2 of the Fundamental Theorem. ◂

SECTION 5.3 EXERCISES

Review Questions

1. Suppose A is an area function of f. What is the relationship between f and A?
2. Suppose F is an antiderivative of f and A is an area function of f. What is the relationship between F and A?
3. Explain in words and write mathematically how the Fundamental Theorem of Calculus is used to evaluate definite integrals.
4. Let $f(x) = c$, where c is a positive constant. Explain why an area function of f is an increasing function.
5. The linear function $f(x) = 3 - x$ is decreasing on the interval $[0, 3]$. Is its area function on the interval $[0, 3]$ increasing or decreasing? Draw a picture and explain.
6. Evaluate $\int_0^2 3x^2\,dx$ and $\int_{-2}^2 3x^2\,dx$.
7. Explain in words and express mathematically the inverse relationship between differentiation and integration as given by the Fundamental Theorem of Calculus.
8. Why can the constant of integration be omitted from the antiderivative when evaluating a definite integral?

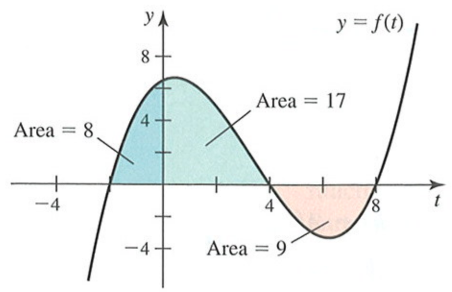
9. Evaluate $\frac{d}{dx}\int_a^x f(t)\,dt$ and $\frac{d}{dx}\int_a^b f(t)\,dt$, where a and b are constants.

10. Explain why $\int_a^b f'(x)\,dx = f(b) - f(a)$.

Basic Skills

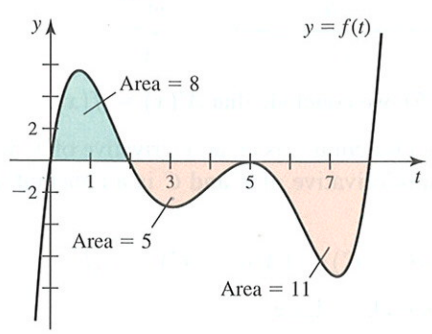
11. Area functions The graph of f is shown in the figure. Let $A(x) = \int_{-2}^x f(t)\,dt$ and $F(x) = \int_4^x f(t)\,dt$ be two area functions for f. Evaluate the following area functions.

a. $A(-2)$ **b.** $F(8)$ **c.** $A(4)$ **d.** $F(4)$ **e.** $A(8)$



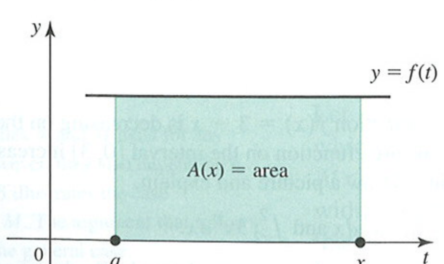
12. Area functions The graph of f is shown in the figure. Let $A(x) = \int_0^x f(t)\,dt$ and $F(x) = \int_2^x f(t)\,dt$ be two area functions for f. Evaluate the following area functions.

a. $A(2)$ **b.** $F(5)$ **c.** $A(0)$ **d.** $F(8)$ **e.** $A(8)$
f. $A(5)$ **g.** $F(2)$



13–16. Area functions for constant functions *Consider the following functions f and real numbers a (see figure).*

a. *Find and graph the area function* $A(x) = \int_a^x f(t)\,dt$ *for f.*
b. *Verify that* $A'(x) = f(x)$.



13. $f(t) = 5,\ a = 0$ **14.** $f(t) = 10,\ a = 4$

15. $f(t) = 5,\ a = -5$ **16.** $f(t) = 2,\ a = -3$

17. Area functions for the same linear function Let $f(t) = t$ and consider the two area functions $A(x) = \int_0^x f(t)\,dt$ and $F(x) = \int_2^x f(t)\,dt$.

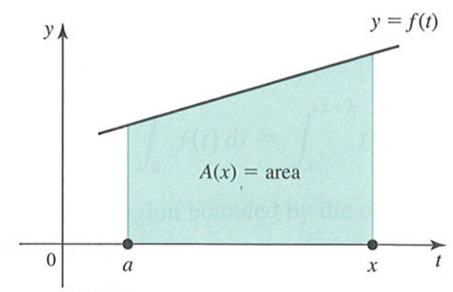
a. Evaluate $A(2)$ and $A(4)$. Then use geometry to find an expression for $A(x)$ for all $x \ge 0$.
b. Evaluate $F(4)$ and $F(6)$. Then use geometry to find an expression for $F(x)$ for all $x \ge 2$.
c. Show that $A(x) - F(x)$ is a constant.

18. Area functions for the same linear function Let $f(t) = 2t - 2$ and consider the two area functions $A(x) = \int_1^x f(t)\,dt$ and $F(x) = \int_4^x f(t)\,dt$.

a. Evaluate $A(2)$ and $A(3)$. Then use geometry to find an expression for $A(x)$ for all $x \ge 1$.
b. Evaluate $F(5)$ and $F(6)$. Then use geometry to find an expression for $F(x)$ for all $x \ge 4$.
c. Show that $A(x) - F(x)$ is a constant.

19–22. Area functions for linear functions *Consider the following functions f and real numbers a (see figure).*

a. *Find and graph the area function* $A(x) = \int_a^x f(t)\,dt$.
b. *Verify that* $A'(x) = f(x)$.

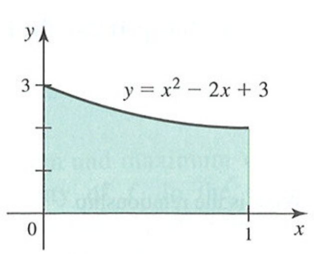


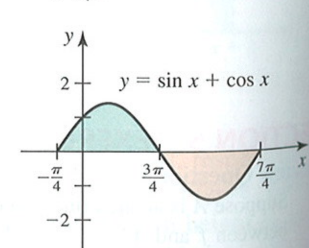
19. $f(t) = t + 5,\ a = -5$ **20.** $f(t) = 2t + 5,\ a = 0$

21. $f(t) = 3t + 1,\ a = 2$ **22.** $f(t) = 4t + 2,\ a = 0$

23–24. Definite integrals *Evaluate the following integrals using the Fundamental Theorem of Calculus. Discuss whether your result is consistent with the figure.*

23. $\int_0^1 (x^2 - 2x + 3)\,dx$ **24.** $\int_{-\pi/4}^{7\pi/4} (\sin x + \cos x)\,dx$





25–30. Definite integrals *Evaluate the following integrals using the Fundamental Theorem of Calculus. Sketch the graph of the integrand and shade the region whose net area you have found.*

25. $\int_1^4 (1 - x)(x - 4)\,dx$ **26.** $\int_0^\pi (1 - \sin x)\,dx$

27. $\int_{-2}^3 (x^2 - x - 6)\,dx$ **28.** $\int_0^1 (x - \sqrt{x})\,dx$

29. $\int_0^5 (x^2 - 9)\,dx$ **30.** $\int_{1/2}^2 \left(1 - \frac{1}{x^2}\right)dx$

31–40. Definite integrals *Evaluate the following integrals using the Fundamental Theorem of Calculus.*

31. $\int_{-2}^2 (x^2 - 4)\,dx$ **32.** $\int_0^{\ln 8} e^x\,dx$

33. $\int_{1/2}^1 (x^{-3} - 8)\,dx$ **34.** $\int_0^4 x(x - 2)(x - 4)\,dx$

35. $\int_0^{\pi/4} \sec^2\theta\,d\theta$ **36.** $\int_0^{1/2} \frac{dx}{\sqrt{1 - x^2}}$

37. $\int_{-2}^{-1} x^{-3}\,dx$ **38.** $\int_{-\pi/2}^{\pi/2} (\cos x - 1)\,dx$

39. $\int_1^2 \frac{3}{t}\,dt$ **40.** $\int_4^9 \frac{x - \sqrt{x}}{x^3}\,dx$

41–44. Areas *Find (i) the net area and (ii) the area of the following regions. Graph the function and indicate the region in question.*

41. The region bounded by $y = x^{1/2}$ and the x-axis between $x = 1$ and $x = 4$

42. The region above the x-axis bounded by $y = 4 - x^2$

43. The region below the x-axis bounded by $y = x^4 - 16$

44. The region bounded by $y = 6\cos x$ and the x-axis between $x = -\pi/2$ and $x = \pi$

45–50. Areas of regions *Find the area of the region R bounded by the graph of f and the x-axis on the given interval. Graph f and the region R.*

45. $f(x) = x^2 - 25;\ [2, 4]$ **46.** $f(x) = x^3 - 1;\ [-1, 2]$

47. $f(x) = \frac{1}{x};\ [-2, -1]$

48. $f(x) = x(x + 1)(x - 2);\ [-1, 2]$

49. $f(x) = \sin x;\ [-\pi/4, 3\pi/4]$

50. $f(x) = \cos x;\ [\pi/2, \pi]$

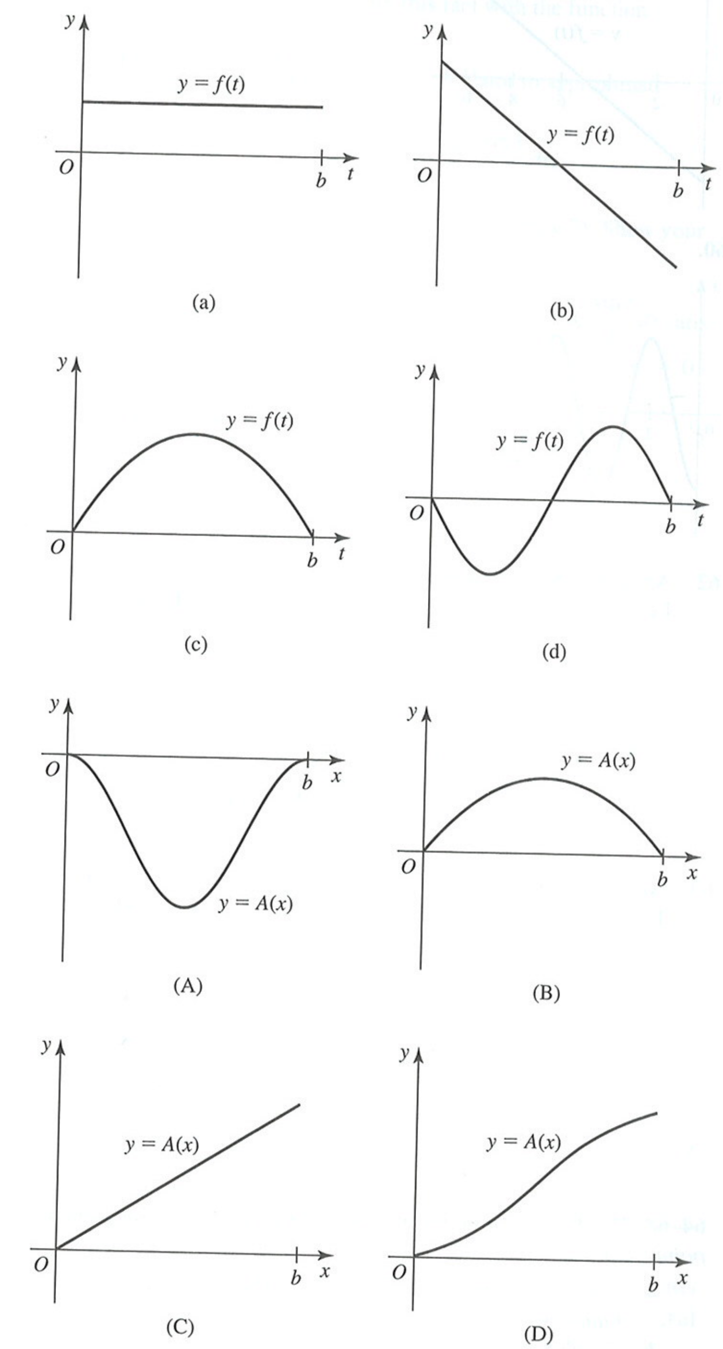
51–56. Derivatives of integrals *Simplify the following expressions.*

51. $\frac{d}{dx}\int_3^x (t^2 + t + 1)\,dt$ **52.** $\frac{d}{dx}\int_0^x e^t\,dt$

53. $\frac{d}{dx}\int_2^{x^3} \frac{dp}{p^2}$ **54.** $\frac{d}{dx}\int_{x^2}^{10} \frac{dz}{z^2 + 1}$

55. $\frac{d}{dx}\int_x^1 \sqrt{t^4 + 1}\,dt$ **56.** $\frac{d}{dx}\int_x^0 \frac{dp}{p^2 + 1}$

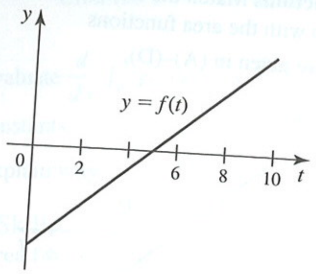
57. Matching functions with area functions Match the functions f whose graphs are given in (a)–(d) with the area functions $A(x) = \int_0^x f(t)\,dt$, whose graphs are given in (A)–(D).



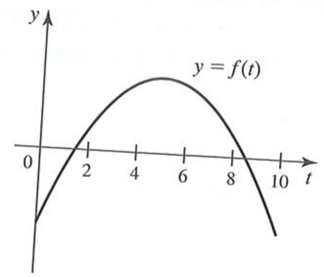
58–61. Working with area functions *Consider the following graphs of functions f.*

a. *Estimate the zeros of the area function* $A(x) = \int_0^x f(t)\,dt$ *for* $0 \le x \le 10$.
b. *Estimate the points (if any) at which A has a local maximum or minimum.*
c. *Sketch a rough graph of A for* $0 \le x \le 10$ *without a scale on the y-axis.*

58.



59.



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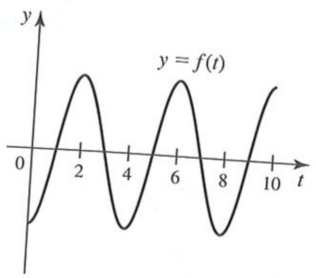

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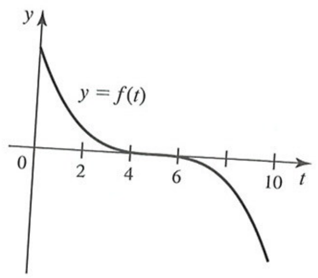
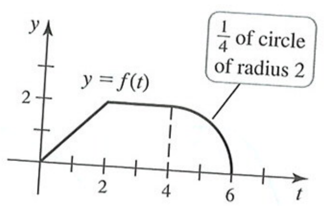
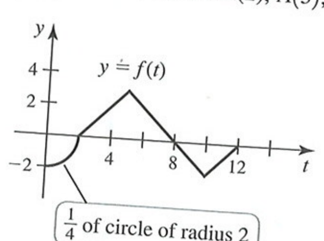


62. Area functions from graphs The graph of f is given in the figure. Let $A(x) = \int_0^x f(t)\,dt$ and evaluate $A(1)$, $A(2)$, $A(4)$, and $A(6)$.



63. Area functions from graphs The graph of f is given in the figure. Let $A(x) = \int_0^x f(t)\,dt$ and evaluate $A(2)$, $A(5)$, $A(8)$, and $A(12)$.



64–68. Working with area functions *Consider the function f and the points a, b, and c.*

a. *Find the area function $A(x) = \int_a^x f(t)\,dt$ using the Fundamental Theorem.*
b. *Graph f and A.*
c. *Evaluate $A(b)$ and $A(c)$ and interpret the results using the graphs of part (b).*

64. $f(x) = \sin x$; $a = 0, b = \pi/2, c = \pi$

65. $f(x) = e^x$; $a = 0, b = \ln 2, c = \ln 4$

66. $f(x) = x^3 + 1$; $a = 0, b = 2, c = 3$

67. $f(x) = x^{1/2}$; $a = 1, b = 4, c = 9$

68. $f(x) = 1/x$; $a = 1, b = 4, c = 6$

T **69–72. Functions defined by integrals** *Consider the function g, which is given in terms of a definite integral with a variable upper limit.*

a. *Graph the integrand.*
b. *Calculate $g'(x)$.*
c. *Graph g, showing all of your work and reasoning.*

69. $g(x) = \int_0^x \sin^2 t\,dt$ **70.** $g(x) = \int_0^x (t^2 + 1)\,dt$

71. $g(x) = \int_0^x \sin(\pi t^2)\,dt$ (a Fresnel integral)

72. $g(x) = \int_0^x \cos(\pi\sqrt{t})\,dt$

Further Explorations

73. Explain why or why not Determine whether the following statements are true and give an explanation or counterexample.

a. Suppose that f is a positive decreasing function for $x > 0$. Then the area function $A(x) = \int_0^x f(t)\,dt$ is an increasing function of x.
b. Suppose that f is a negative increasing function for $x > 0$. Then the area function $A(x) = \int_0^x f(t)\,dt$ is a decreasing function of x.
c. The functions $p(x) = \sin 3x$ and $q(x) = 4\sin 3x$ are antiderivatives of the same function.
d. If $A(x) = 3x^2 - x + 2$ is an area function for f, then $B(x) = 3x^2 - x$ is also an area function for f.

74–82. Definite integrals *Evaluate the following definite integrals using the Fundamental Theorem of Calculus.*

74. $\frac{1}{2}\int_0^{\ln 2} e^x\,dx$ **75.** $\int_1^4 \frac{x-2}{\sqrt{x}}\,dx$ **76.** $\int_1^2 \left(\frac{2}{s} - \frac{4}{s^3}\right)ds$

77. $\int_0^{\pi/3} \sec x \tan x\,dx$ **78.** $\int_{\pi/4}^{\pi/2} \csc^2\theta\,d\theta$ **79.** $\int_1^8 \sqrt[3]{y}\,dy$

80. $\int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2-1}}$ **81.** $\int_1^2 \frac{z^2+4}{z}\,dz$ **82.** $\int_0^{\sqrt{3}} \frac{dx}{1+x^2}$

T **83–86. Areas of regions** *Find the area of the region R bounded by the graph of f and the x-axis on the given interval. Graph f and show the region R.*

83. $f(x) = 2 - |x|$; $[-2, 4]$

84. $f(x) = (1 - x^2)^{-1/2}$; $[-1/2, \sqrt{3}/2]$

85. $f(x) = x^4 - 4$; $[1, 4]$ **86.** $f(x) = x^2(x - 2)$; $[-1, 3]$

87–90. Derivatives and integrals *Simplify the given expressions. Assume that derivatives are continuous on the interval of integration.*

87. $\int_3^8 f'(t)\,dt$ **88.** $\frac{d}{dx}\int_0^{x^2} \frac{1}{t^2+4}\,dt$

89. $\frac{d}{dx}\int_0^{\cos x} (t^4 + 6)\,dt$ **90.** $\frac{d}{dx}\int_x^1 e^{t^2}\,dt$

Additional Exercises

91. Zero net area Consider the function $f(x) = x^2 - 4x$.

a. Graph f on the interval $x \geq 0$.
b. For what value of $b > 0$ is $\int_0^b f(x)\,dx = 0$?
c. In general, for the function $f(x) = x^2 - ax$, where $a > 0$, for what value of $b > 0$ (as a function of a) is $\int_0^b f(x)\,dx = 0$?

92. Cubic zero net area Consider the graph of the cubic $y = x(x - a)(x - b)$ where $0 < a < b$. Verify that the graph bounds a region above the x-axis for $0 < x < a$ and bounds a region below the x-axis for $a < x < b$. What is the relationship between a and b if the areas of these two regions are equal?

93. Maximum net area What value of $b > -1$ maximizes the integral

$$\int_{-1}^b x^2(3 - x)\,dx?$$

T **94. Maximum net area** Graph the function $f(x) = 8 + 2x - x^2$ and determine the values of a and b that maximize the value of the integral

$$\int_a^b (8 + 2x - x^2)\,dx.$$

95. An integral equation Use the Fundamental Theorem of Calculus, Part 1, to find the function f that satisfies the equation

$$\int_0^x f(t)\,dt = 2\cos x + 3x + 2.$$

96. Max/min of area functions Suppose f is continuous on $[0, \infty)$ and $A(x)$ is the net area bounded by the graph of f and the t-axis on $[0, x]$. Show that the maxima and minima of A occur at the zeros of f. Verify this fact with the function $f(x) = x^2 - 10x$.

T **97. Asymptote of sine integral** Use a calculator to approximate

$$\lim_{x\to\infty} S(x) = \lim_{x\to\infty}\int_0^x \frac{\sin t}{t}\,dt,$$

where S is the sine integral function (see Example 7). Show your work and describe your reasoning.

98. Sine integral Show that the sine integral $S(x) = \int_0^x \frac{\sin t}{t}\,dt$ satisfies the (differential) equation $xS'(x) + 2S''(x) + xS'''(x) = 0$.

99. Fresnel integral Show that the Fresnel integral $S(x) = \int_0^x \sin(t^2)\,dt$ satisfies the (differential) equation

$$(S'(x))^2 + \left(\frac{S''(x)}{2x}\right)^2 = 1.$$

100. Variable integration limits Evaluate $\frac{d}{dx}\int_{-x}^x (t^2 + t)\,dt$. (*Hint:* Separate the integral into two pieces.)

QUICK CHECK ANSWERS

1. $0, -35$ **2.** $A(6) = 44$; $A(10) = 120$ **3.** $\frac{2}{3} - \frac{1}{2} = \frac{1}{6}$ **4.** If f is differentiated, we get f'. Thus f is an antiderivative of f'.

5.4 Working with Integrals

With the Fundamental Theorem of Calculus in hand, we may begin an investigation of integration and its applications. In this section we discuss the role of symmetry in integrals, use the slice-and-sum strategy to define the average value of a function, and then explore a theoretical result called the Mean Value Theorem for integrals.

Integrating Even and Odd Functions

Symmetry appears throughout mathematics in many different forms, and its use often leads to insights and efficiencies. Here we use the symmetry of a function to simplify integral calculations.

Section 1.1 introduced the symmetry of even and odd functions. An **even function** satisfies the property that $f(-x) = f(x)$, which means that its graph is symmetric about the y-axis (Figure 5.50a). Examples of even functions are $f(x) = \cos x$ and $f(x) = x^n$, where n is an even integer. An **odd function** satisfies the property that $f(-x) = -f(x)$, which means that its graph is symmetric about the origin (Figure 5.50b). Examples of odd functions are $f(x) = \sin x$ and $f(x) = x^n$, where n is an odd integer.

Special things happen when we integrate even and odd functions on intervals centered at the origin. First, suppose f is an even function and consider $\int_{-a}^a f(x)\,dx$. From